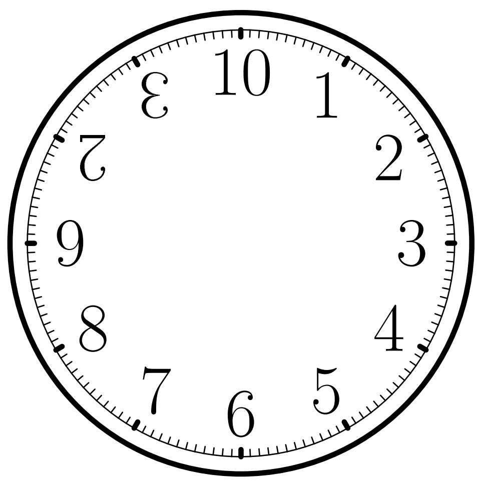
A Primer

on

Dozenalism



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Contents

Figures and Tables

1 Introduction

Man deals with numbers every day, and has since the very beginning. It is not hard to imagine Adam and Eve counting; if nothing else, they surely counted the number of children they had. But we can be equally sure that they counted the fruits which they plucked from the trees in the Garden, and the number of branches which needed to be pruned, and the quantity of vegetables and herbs which were available for their use. They must have counted any number of things, and obviously they used numbers when they did.

It is further certain the Adam and Eve engaged not only in counting, but in calculation. When Adam went forth into the garden to select his food, he surely looked at the number of fruits on a given tree to determine how many would remain after he had his fill. And is this not subtraction? When he was gathering up what he needed to take back to Eve, he surely calculated how many fruits he would need to add to what he already had until he would have enough. And is this not addition?

After the Fall, of course, counting and calculation continued, and as man's society expanded it grew ever more complex. Cain and Abel must have determined how much of the fruit of their labors they could spare for sacrifice to the Lord God; clearly, their calculations mattered significantly, as Abel's calculation, in terms of quality and quantity, was certainly better than Cain's. And when Cain left subject to his punishment for killing his brother, is there any doubt that he continued to engage in calculation on a daily basis, determing what he would need, how many people were in a given area, how much food would be required to feed his household, and so on *ad infinitum*?

In our own day, numbers are everywhere. Everyone with even a modicum of financial responsibility deals daily with calculations, and often complex ones, concerning the amount of money he has and how much he can afford to spend. Sums and differences (that is, the results of addition and subtraction), of course, are simple and routine; frequently we must also engage in multiplication and division, often of complex sorts. Our homes are nearly universally subject to mortgages, which involve calculations of interest payments, and sometimes are complicated further by the addition of unpaid interest to the principle (what financiers call the *capitalization* of interest). We multiply the widths and lengths of rooms to determine square footage, and of walls to determine how much paint or wallpaper will be needed. We divide the amount of food our pets consume daily by the quantity of food in the bag to determine when we need to buy more. The subjects and frequency of our dealing with numbers are truly incalculable; they are a daily, and even hourly, inescapable part of our lives.

So why an article about numbers? Why embark upon a laborious explanation of something that is so ubiquitous? Surely, since they are everywhere, we all must understand numbers? In fact, however, most people have only the vaguest understanding of numbers, if any. We know the rules for manipulating them, and we have some intuitive idea of what they represent; but we have little idea of what they are and how they work. Because of this, we have little if any idea about how they *ought* to work and whether our system is as good as it could be.

This article attempts to provide an easy and accessible guide to numbers and how they work. As such, it attempts to answer three major questions:

1. What are numbers?

2. How do they work?

3. Could the numbers we use be and work better than they are and do?

These are questions that we in our society are not trained to ask, and therefore examining them will probably stretch our minds. I ask the reader to bear with me. It will undoubtedly involve some effort to understand our subject. However, given the ubiquity and importance of numbers in our daily lives, I can assure the reader that this effort will be richly rewarded throughout, from the beginning to the end.

Numbers are far from the exclusive domain of mathematicians. We use numbers so much that we should *all* be interested in them, whether we enjoy mathematics or not. For this reason, devising and using a sensible number system is in the interest and to the benefit of everyone, from the youngest schoolchild to the most grizzled of grandfathers. And with that, we embark on our journey, a new "Excursion in Numbers."[1] I hope and pray that this journey will be interesting and profitable to all who venture on it with me.

2 The Nature of Numbers

As explained briefly above,[2] numbers are everywhere, and we use them constantly. But very few of us really know what we're dealing with when we're handling them. So our first task is to ask and answer the question, what exactly is a number?

The mathematicians will answer this with, "Lots of things." They will tell you about natural numbers, whole numbers, rational numbers, irrational numbers, even *imaginary* numbers. And all of these types of numbers have their uses. We will talk about those things, too, in very simple terms. However, for now let's stick with the most basic question: what are numbers? You know, those things that we use when we're counting something or doing calculations on real things?

Let us answer this question by first examining the most fundamental task that we perform with numbers: counting. When we begin to count some group of things—say, the number of apples in a basket—we begin with one, then proceed to two, then to three, and so on. When we have reached the last apple, we stop, and we've finished counting. There is no limit to how many apples we can possibly count; in theory, we could keep counting forever and never run out of numbers. But there *is* a *minimum* number of apples we can count: one.

What about the apples are we saying when we count them? We're saying that there is a given *quantity* of apples. However, we can also think about quantity even apart from the apples; for example, we can think about "eleven" considered simply as a number, not only as eleven apples, eleven cars, or eleven something else. So fundamentally, number is *quantity con-*

[1] F. Emerson Andrews, *An Excursion in Numbers* (The Dozenal Society of America), in THE ATLANTIC MONTHLY, October 1152.

[2] *See supra*, Section 1, at 3.

sidered as quantity. And that is indeed what number is, at its most basic level.

Mathematicians refer to such numbers as *natural* numbers; that is, the numbers from one to infinity. (Mathematicians define natural numbers as a *set*; they write this set as $\{1, 2, 3, \ldots\}$.)

Other, more complex forms of numbers can be derived from this. For example, sometimes it is helpful for us to imagine a total lack of quantity; we call that "zero," and write it "0." Indeed, we can even imagine numbers that represent "quantities" *less* than the lack of quantity. For example, when we subtract three from seven, we get 4, like so:

$$7 - 3 = 4 \tag{1}$$

However, let's extend the idea of subtraction (which is easy to understand) and switch those numbers around, subtracting seven from three.

$$3 - 7 = -4 \tag{2}$$

Naturally speaking, equation 2 really doesn't mean anything sensible at all. In terms of actual quantity, you can never have less than zero apples. However, it is very helpful, when dealing with more complex calculations, to be able to use "numbers" that don't exist in the real world. Thus, we imagine negative numbers, numbers less than zero, so that we can work with them and make our other calculations easier. The set of numbers which contains zero and the natural numbers are called *whole* numbers; the set is written $\{0, 1, 2, 3, \ldots\}$. The set of numbers which contains zero and numbers less than zero, as well as the natural numbers we described earlier, are called by mathematicians *real* numbers, and the set is written $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.[3]

Then, of course, there are fractions (what in our current number system we often, inaccurately, call "decimals"). Fractions are not properly called "numbers," in the literal sense; after all, when we cut an apple in half, we do not have two apples. However, treating them as numbers simplifies our mathematics immensely. Fractions are, purely and simply, parts of a whole. They can be written either as parts per a whole—for example, as $\frac{1}{2}$ for one part of two, or $\frac{2}{3}$ for two parts out of three—or as "decimals," utilizing the system of place notation[4]—as in 0.5 for one half, or 0.25 for one quarter.

[3]The set also contains all fractional parts of numbers, whether rational or irrational, but we need not trouble ourselves with that at this time.

[4]For an explanation of this term, *see infra*, section 3.1, at 8.

Fractions written as fractions—that is, writing two-thirds as $\frac{2}{3}$—are always rational. However, fractions written with place notation[5] can be either *rational* or *irrational*. A fraction written with place notation is *rational* if it can be written with perfect accuracy in a finite number of digits. An example is $\frac{1}{4}$, which can be written with perfect accuracy in only two digits, 0.25. A fraction is *irrational* if it cannot be so written. An example is $\frac{1}{3}$, which must be written as a zero, a "decimal point," and an infinite number of threes. As another example, "pi," or π (which in mathematical terms is defined as the ratio of the circumference of a circle to its diameter), is 3.14159 and so on, *ad infinitum*, with no discernible pattern of digits whatsoever.

Finally, there are *imaginary* numbers, which are numbers which do not and cannot exist, but which, if we pretend that they do exist, make certain very complex mathematics easier to do. As any elementary student can tell you, it is impossible to find the square root of a negative number. This is because multiplication can only produce a negative number if the signs are mixed; that is, if a negative number is multiplied by a positive number, or vice versa. If two negative numbers or two positive numbers are multiplied, the result will be positive. That means that you can *never* multiply a number by itself and get a negative number; it's simply impossible. Therefore, negative numbers cannot have square roots (which is defined as the number which, when multiplied by itself, produces the first number.) However, mathematicians have found that if they assign a letter, i, to mean "the square root of -1," they can do some extremely complex mathematics much, much more easily. While this is an interesting device of mathematics, and I wanted to make you aware of its existence, it won't concern us at all in the tasks ahead, so don't worry too much about it unless you're interested.

So, as we have seen, there are many different meanings for the word "number." For our purposes, however, number is just anything designed to refer to quantity or to measurement. That much is simple enough.

However, why do we *write* numbers the way we do? That is, why do we write "three" as 3, and "twenty-seven" as 27, and not as III and XXVII? That is where these simple questions began to get interesting, and that is the subject of our next section.

[5] *Id.*

3 Possible Systems of Numbering

We currently write numbers according to the *Arabic* system, though it is more properly called *Indo-Arabic*,[6] and we do so with a base number of ten.[7] But is this the only, or even the best, way for us to write numbers? What are some other ways that we could write numbers, and which of them is easiest and best?

3.1 Systems of Notation

Possible ways of writing numbers I will refer to as *systems of notation*, and there are many possible systems of notation which are or have been in use throughout the world. These may make use of place notation, of varying bases, or of any number of other conceivable ideas.

The system of numbering most people in the West are familiar with, besides the common Indo-Arabic system, is commonly called *Roman numerals*. Roman numerals are far from crude or simple; they were good enough for the largest and richest empire in the world for over a thousand years, and for the rest of Europe even after that. Roman numerals make use of a system I will refer to as *varying bases* in order write numbers sensibly.

Here's how they work. The Romans selected various numbers which they considered important enough to deserve their own symbols; these are the numbers which I call, for lack of a better term, *bases*. The first special, important number is, obviously enough, one, which they wrote as I. The next is five, written as V. Then, they continued to select numbers they deemed important, at larger and larger intervals from the last: ten, X; fifty, L; one hundred, C; five hundred, D; one thousand, M; and so on. (Five thousand was sometimes represented by a D with a line over top of it, $\bar{\mathrm{D}}$; however, I have never seen a symbol larger than that, though there may well have been such.) These few symbols—only seven or eight, fewer than the ten symbols we currently have—were sufficient for the Romans to write any number they ever had need of.

But how did they write the numbers between their special, important ones? For example, how did they write four, or seventy-eight? They did so by simply reduplicating the lower symbols until the total added up to the

[6]For more on this topic, *see infra*, section 3.1, at 8.
[7]For more on bases, *see infra*, section 3.2, at 11.

number they wanted. (Later Roman numerals also made use of subtraction, as in "IV," but we'll stick with the original system for now.) To write four, Romans just added ones together until they equalled four, like so: IIII. The next number was five, which is one of the special numbers, so it was written with its own symbol: V. Six, of course, is not a special number, so they again combined the special numbers until it equalled six: VI. All numbers were done in this way. So, for example, we arrive at seventy-eight, LXXVIII, by adding the special numbers together:

$$\mathrm{L}(50) + \mathrm{X}(60) + \mathrm{X}(70) + \mathrm{V}(75) + \mathrm{I}(76) + \mathrm{I}(77) + \mathrm{I}(78) = \mathrm{LXXVIII} \quad (3)$$

This system is perfectly adequate for representing all conceivable natural numbers; for example, the current year, 2009, is written simply as MMVIIII (or MMIX in the later notation, using subtraction), and the year 1883 is written as MDCCCLXXXIII.

However, Roman numerals are very difficult to do calculations with, the way that elementary students do so easily with our current common system. Adding with Roman numerals is not too complicated; one simply combines all the symbols together, then adds them together to make larger ones when possible, like the following equation, which adds thirty-seven to forty-two to yield a result of seventy-nine:

$$\mathrm{XXXVII} + \mathrm{XXXXII} = \mathrm{XXXXXXXVIIII} = \mathrm{LXXVIIII} \quad (4)$$

As you can see, first one puts all the symbols from both numbers together, yielding the monstrous XXXXXXXVIIII. Then, one counts out the Xs until one has enough to make an L (or a C, or whatever the highest symbol one can make with them might be; in this case, it is an L). One removes the five Xs which make the L, then inserts the L, yielding LXXVIIII. One does the same with the other symbols (in this case, no other symbols require it), and then one has the result. Subtraction is done the same way, but backwards, though it can get considerably more difficult than addition.

When one comes to do multiplication in Roman numerals, one sees just how hopeless the system really is. Multiplication, in simple terms, is simply repeated addition. For example, when one multiplies seven by three, one is really adding seven to itself three times, like so:

$$7 \cdot 3 = 7 + 7 + 7 = 21 \quad (5)$$

Truly, this is the height of simplicity. We all know how to multiply numbers by hand, working out each step, in our current system, so I won't demonstrate it. But let's try to do it in Roman numerals.

$$\mathrm{VII} \cdot \mathrm{III} = \mathrm{VII} + \mathrm{VII} + \mathrm{VII} = \mathrm{VVVIIIIII} = \mathrm{XVVI} = \mathrm{XXI} \qquad (6)$$

Not only is this an extremely simple example, but I also skipped a step in order to save space on the line (VVVIIIIII should have become XVIIIIII before becoming XVVI). Difficult, of course, but still doable. Now, however, try multiplying MDCCCLXXXIII by MCCCXXVI. The complexity problem quite rapidly spins out of control.

So a simpler means, if one exists, is desirable.

That simpler means is called *place notation.* Place notation begins by assigning a certain finite number of symbols to some of the lowest numbers. In our system, we have nine symbols, for the numbers 1, 2, 3, 4, 5, 6, 7, 8, and 9. However, it then invents another number, which we call *zero* and write as 0. That zero is what makes place notation possible; for with place notation, the location of a digit is what determines its value, and without zero one could not ensure that a digit was in its proper place.

Let's look at an example. The year 1896 is the year that my great-grandfather was born. We write that year by the number of years that have passed since the birth of Christ; in this case, one thousand, eight hundred and ninety-six. We see four digits written there, a one, an eight, a nine, and a six. The one in this number, however, does not mean the number one. Rather, it means the number one *thousand.* Similar, the eight does not mean simply eight, but rather eight *hundred.* The nine does not mean nine, but nine *tens.* The six, however, means exactly what it looks like: six.

How can we tell what each digit is supposed to mean? Each one looks precisely the same whether it means six *thousand* or only six. However, we can still tell exactly what each digit means based on where in the number it is placed; hence, the name *place notation.*

The number furthest to the right (absent a decimal point, of course) means only what it says. In this case, it means six; that is, six units of one. The number immediately to the left of it, however, doesn't mean simply itself, nine. Rather, it means nine units of ten. The number to the right of that, eight, doesn't mean simply eight. Rather, it means eight units of *ten times ten*, or eight units of one hundred. Finally, the one doesn't mean simply one, but rather one unit of *ten times ten times ten*, or one unit of one thousand. We can only tell what the whole number means once we determine what each digit means, as we just did, and then add them all together; in this case, the result of doing these calculations is one thousand, eight hundred and ninety-six.

Put more precisely, for each place in a number we must consider two facts: first, what multiple of ten belongs there, and second the value of the digit that's in that place. We start with the digit furthest to the right, except those to the right of the decimal point. In this case, that digit is 6, which has the value of six. Since this is the first digit, we multiply six by one, which yields a result of six. We then save that six to be added to the total later. Moving to the left, we see that the digit occupying that place is 9, with a value of nine. We multiply the value of the first digit's place, one, by ten, yielding an answer of ten. So we know that the value of this place in the number is ten. We then multiply ten by the value of the digit in that place, which is nine, and get a result of ninety. We then put that ninety with the six to be added to the total when we're done. Next, we move to the third place, which is occupied by 8, with a value of eight. The number that belongs in the third place is ten multiplied by the value of the last place, which was ten (remember, the place occupied by the nine?), yielding a result of one hundred. So we know that the third place in the number is the one that measures hundreds. The digit here is 8, so we multiply eight by one hundred, getting eight hundred. We put the eight hundred with the ninety and the six to be added to the total at the end. Finally, we come to the last digit, 1, with a value of one. We then multiply the last place's value, one hundred, by ten, making a thousand. So we know that the fourth place is the one that counts the thousands in the number. We then multiply a thousand by the digit in that place, one, and get one thousand. Now, we're out of digits, so we put the one thousand with the eight hundred, the ninety, and the six, and we add them up, giving us one thousand, eight hundred and ninety-six.

This is really quite ingenious, and allows for easy expansion of numbers. If, for example, we add another digit to the left, making the number 71,896, we know precisely what the seven means, too; it means seven units of ten times one thousand; that is, seven units of ten thousand, or seventy thousand. Then, we simply add all the numbers up—seventy thousand, one thousand, eight hundred, ninety, and six—and we have our final number.

But what if there are no units of some multiple of ten? For example, what if our number were not one thousand, eight hundred and ninety-six,

Place Notation				
Digits	4	6	7	8
Multiples of Ten	1,000	100	10	1
Values	4,000	600	70	8
Total				4,678

Table 1: A table calculating four thousand, six hundred and seventy-eight using place notation in base ten.

but rather one thousand and ninety-six? Doesn't this place notation system break down at that point? Yes, it does; that's why it was not used for the vast majority of human history. However, place notation has a saving grace: the digit that means nothing, 0. When there are no units of a given multiple of ten—in this case, there are no units of one hundred, which is ten times ten—simply put a 0 in that place. So, one thousand and ninety-six is written simply as 1096; that is, one unit of ten times ten times ten, or one thousand; zero units of ten times ten, or one hundred; nine units of ten, or ninety; and six units of one, or six. Without zero, of course, this system does not work; with zero, however, it can write all possible numbers with no ambiguity.

Furthermore, Roman numerals cannot be used to write fractions at all; however, place notation makes writing fractions easy. One does this simply by adding digits to the *right*, rather than to the left. When one adds digits to the left, one is adding units of *increasing* multiples; when one adds them to the right, one is adding units of *decreasing* multiples. One we introduce a symbol to separate the whole units from the fractional ones, we have no problems at all. So, for example, one can write 7.2, which means seven units of one, plus two units of one divided by ten, or two-tenths ($\frac{2}{10}$). Adding another digit to the right, one is adding units of one divided by ten divided by ten, or one-hundredth ($\frac{1}{100}$). And so on; one simply multiplies the digit by the appropriate multiple of ten, adds all the results together, and one has the number, including its fractional parts.

All this laborious exercise of multiples, multiplication, and final addition happens automatically in our minds, of course, because we are extremely familiar with this sytem and have been using it extensively all our lives. This complicated process of multiplication and division is, however, precisely what we're doing every time we read a number written in our current, common system, though we do it very quickly and without much thought. Imagine, however, someone who had been working only with Roman numerals for decades first encountering this system. A digit which means nothing? Seriously? And do you really expect me to engage in all this multiplication and division, then add the results together, just to read a simple number?

And yet, with appropriate practice, the numbers are just as easy to read as Roman numerals are, and are immeasurably easier to use in calculations, even calculations as simple as addition and multiplication. We are all familiar with the way that such operations are done using our number system, so I'll refrain from giving examples; however, it's easy to see that our way is significantly easier than that one must use if writing in Roman numerals.

And so we can clearly see that place notation is by far the superior system for writing numbers, at least of those that have been tried so far. The astute reader will notice, however, that the number ten seemed extremely prominent in our discussion of place notation. Why the number ten? That's an excellent question; the number ten is so significant because it is the *base* of our current system, and that forms the subject of the next section.

3.2 The Concept of the Numerical Base

Place notation is clearly an excellent way of writing numbers. However, it does present us with a choice that other number systems do not: namely, we have to choose a *base* for our notation.

We've already seen how, in our current system, adding digits to the left of the equation adds units of the next multiple of ten; that is, adding a 6 to the end of the number 56 adds not another six, but rather another six *hundred* to the total number. This is because our current number system uses ten as its base. Thus, the number written 656 involves six units of one, plus five units of ten times one, or ten, and then six units of ten times ten, or hundred, making a total of six hundred and fifty-six.

Many people confuse the benefits of the place notation system with the benefits of the number ten. That is, they equate the two, and believe that our easy system of numbering is not due to the brilliance of the digit zero or the way that new places add or subtract multiples, but rather due to the fact that it's based on the number ten. This impression, however, is simply not true.

Place notation works perfectly and easily with *any* base, whether it's ten, five, or fifty. Indeed, the ancient Mayans, who were brilliant mathematicians, used a based of twenty, and the ancient Babylonians, also skilled in the mathematical arts, used a base of sixty (which yielded our system of measuring three hundred and sixty degrees in a circle). Despite not using base ten, these peoples were able to take full advantage of the benefits of using place notation. There really is nothing special about ten in this regard.

What is required for using place notation, if the number ten is not? Only two things:

1. A digit to hold place when there is no unit of a given multiple in a number. Namely, a digit for zero.

2. A total number of digits equal to the base of the system; that is, zero, plus digits for all whole numbers between zero and the base. In our system, these digits are 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9; ten digits, equal to the base of our system, ten.

Our society already uses place notation for several bases other than ten, though mostly these are used by computers. The three most common bases are binary (base two), octal (base eight), and hexadecimal, or hex (base sixteen).

Let's look for a minute at base two, the lowest possible base. To write numbers in base two, one needs two things: a digit to represent zero, and a total number of digits equal to the base itself. Therefore, we need a zero and one other digit. Customarily, we simply use 0 and 1 for these two digits. And that's it; we have our binary number system.

Let's begin by counting in binary. We'll start with zero, and write it thus: 0. After zero, we proceed to one, written thus: 1. Now, we want to count to two. Two, however, is our base; that means we've reached a multiple of our base, and we have to add a digit to the left to properly write the number. So, we write the number two in binary: 10.

Remember to forget your training while you're reading this. Mentally, when we see the digit one followed by the digit zero, we don't read it properly, to mean "one unit of ten and zero units of one." Rather, we read it simply as "ten." However, those digits only mean "ten" when one is writing in base ten; here, we are writing in base two, which means that "10" means something very different.

We know that it means two, since we just counted to it. However, let's look at the 10 and determine how it means two in binary without counting up to it. We start with the rightmost digit (absent any fractional part); this is zero. That means that we have zero units of one. (This digit will mean the same thing in every base; that is, units of one.) Then, we move on to the next rightmost digit, in this case 1. Since we are using base two, that 1 means that we have one unit of our base, which is two. So, we have one unit of two and zero units of one. Two plus zero equals two; therefore, binary 10 means two.

Let's illustrate the concept a little further, with a larger binary number. Let's try 101101. Again, we start with the rightmost digit, absent any fractional part. (We don't have a fractional part here; we could, just as easily as we can in base ten, but we're trying to keep it simple here.) That digit is 1. Therefore, we know that we have one unit of one. Moving on, there is a zero in the next rightmost digit. That means we have zero units of two, which is our base. Moving on, we see that the next digit is 1 again. That means that we have one unit of four, because the next multiple of two is four. Moving on, the next digit is again 1; that means we have one unit of eight, since the next multiple of two is eight. Moving on, the next digit is 0; that means we have zero units of sixteen, since the next multiple of two is sixteen. Finally, the leftmost digit is another 1; that means we have one unit of thirty-two, that being the next multiple of two after sixteen. (That is, sixteen times two is thirty-two.) Adding all of these together—thirty-two, eight, four, and one—we see that binary 101101 is equal to forty-five, written 45 in base ten.

Let's look next at octal, which is used commonly in computers. (Unix file permissions, for example, are often entered directly using octal sums.) Octal is base eight. To write numbers in base eight, we need a zero, plus a total of eight digits. We'll use 0, 1, 2, 3, 4, 5, 6, and 7. Remember, forget your training in base ten; we're using base eight. That means that 10 does *not* mean "ten"; it means eight, nothing more nor less.

The octal number we'll work on will be 462. Remember, this does not mean "four hundred and sixty-two"; we're using base eight. We'll examine a better system for talking about numbers in at least one base later[8]; for now, try just to think of this as "four sixty-fours, six eights, and two ones." (If this seems cumbersome, remember that the French refer to ninety-eight as *quatre-vingt-dix-huit*, which means literally "four twenties, ten, and eight," and they don't seem to have any more trouble than anyone else with mathematics.)

So, as always, we start with the rightmost digit absent any fractional part.

[8] *See infra*, section 4.3.3, at 28.

462 in Base Eight			
Digits	4	6	2
Multiples of Eight (in base ten)	64	8	1
Values (in base ten)	256	48	2
Total (in base ten)			306

Table 2: A table calculating a number in place notation in base eight.

In this case, that digit is 2. We know, therefore, that we have two units of one, and can move on. The next digit is 6; our base, eight, multiplied by one equals eight, so we know that we have six units of eight, which is translated into base ten by simple multiplication:

$$6 \cdot 8 = 48 \tag{7}$$

The next digit is 4; therefore, we know that we also have four units of sixty-four, since the next multiple of eight after eight itself is sixty-four. Translated into base ten, we simply multiply four by sixty-four, yielding two hundred and fifty-six. There are no more digits, so we add our two units of one to our six units of eight (in base ten, forty-eight) to our four units of sixty-four (in base ten, two hundred and fifty-six), yielding a total, in base ten, of three hundred and six.

When we move to hexadecimal, though, we have an additional problem. Our current system provides us with only ten symbols; namely, 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. But with hexadecimal (base sixteen), we need not ten, but *sixteen* symbols. What are we to do? The standard solution is to simply begin using the letters of the alphabet, in standard alphabetical order. Therefore, our set for base sixteen will look like this:

0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F

And after F, of course, will be 10 (which means, remember, not "ten," but "sixteen"). We'll explore a more satisfactory solution for a base intended for daily use later[9]; however, until we decide that such a better solution is even worth finding, we'll utilize this solution for any base higher than ten.

[9] *See infra*, section 4.3.2, at 24.

Since we're pretty familiar at this point with the concept of the base, let's proceed through only one simple example and move on. For that example, let's translate the number 5F7 into decimal, to help us understand what it means, since we're not used to thinking in hexadecimal. First, the rightmost digit (as always, absent any fractional part) is the 7, which indicates seven units of one. The second digit, however, is F; this, remember, indicates the number fifteen, and means that we have fifteen units of the base multiplied by one, or sixteen. Fifteen units of sixteen is, in decimal, 240. So we add 240 to the first digit we analyzed, 7, and get the decimal number 247. The third digit from the right is 5, indicating five units of sixteen times sixteen, or the decimal 256. Five units of 256 is the decimal 1280. So, we add 1280 to our current number, 247, and find that the hexadecimal 5F7 equals the decimal 1527.

Keep in mind that I'm only translating these numbers into base ten for our convenience, because we're accustomed to thinking in base ten. There really is no need to do so. 46 in base eight equals thirty-eight just as much as 38 in base ten does. The 38 seems more natural to us; however, that is only because we're so used to numbers in base ten. Had we grown up using base eight in the same way, we'd be stretching our minds to figure out how 38 could possibly equal the number thirty-eight, and remarking on how much easier 46 is. (Of course, we'd probably speak about numbers differently, as well, saying something like "four eights and six." That, again, is a topic we'll address in detail later.[7])

But isn't base ten still easier? After all, in base ten, when one wants to multiply by ten, one simply moves the decimal place over. $4.50, for example, multiplied by ten is equal to $45.00; it's as easy as pie. However, this sort of property is common to all bases; that is, multiplication or division by the base, or by a multiple of the base, is a simple matter of moving the fractional marker over the appropriate number of digits. Base ten does not have a monopoly on this characteristic.

For example, in base eight, 0.4 is equal to $\frac{1}{2}$; that is, the digit directly to the right of the "decimal" point (we should really, in base eight, call it the octal point) represents units of one-eighth, just as in base ten it represents units of one-tenth. Four units of one-eighth is one-half; therefore, 0.4 in base eight is one-half. Let's say that we want to multiply this number by eight. The solution is to simply move the octal point to the right one place. Rather

[7] *See infra*, section 4.3.3, at 28.

than 0.4, then, we end up with simply 4, four units of one. To divide by eight, we simply move the octal point to the left, giving us 0.04, or four units of sixty-fourths ($\frac{4}{64}$). Base ten certainly has the benefits of such division; however, when using place notation, multiplication or division by the base is *always* easy in this way, no matter what the base happens to be.

By now, we should all be thoroughly familiar with the concept of bases, and realize that the benefits of place notation depend not upon the base used, but upon the nature of place notation itself. We also know that, while in certain limited circumstances people may use bases two, eight, and sixteen, the vast majority of the planet uses base ten the vast majority of the time. However, what base *ought* we to use? Is base ten the best base? Or should we adopt and use some other base for our number system?

4 The Case for Dozens

So we've established that place notation is the best possible system, at least of those that have been tried so far. But is our current base the best possible base?

4.1 Criteria of a Good Base

Before, however, we can answer the question of what would be the best base, we must first ask another: how do we judge how good a base is? When we're considering a base, what criteria do we use to determine whether it's good or bad?

First, we have to take into account the necessary limitations of place notation. Most particularly, we have to remember that we require a number of symbols equal to the value of our base. So, for example, our current base ten system requires ten symbols, zero and one for every whole number less than the base. In our current system, those symbols are 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. In order to limit the number of symbols we'll need, the perfect base would have to be small enough to ensure a manageable number of symbols. This means that bases like eight, ten, or twelve are open for consideration, but a base like forty or sixty is probably too high.

Second, we want to make sure that our base is large enough to ensure that the length of numbers remains reasonable. For example, we've already seen that the number which is represented in base ten by the digits "45" is represented by "101101" in binary; in other words, what requires only two digits in decimal requires three times as many in binary. The base of two is simply too small; when dealing with such small numbers as forty-five, we certainly don't want to have to toss around six digits. Furthermore, imagine working with larger numbers; decimal 71,896 is represented as the incredible "10001100011011000" in binary. Even such a relatively small number as seventy-one thousand has become quite unmanageable. On the other hand, in hexadecimal (base sixteen), the same number is only "118D8," the same number of digits as in base ten. So while we want a base which is small enough to ensure a reasonable number of symbols, we also want one which is large enough to produce reasonable number lengths.

Third, we want to ensure that our base is divisible by as many numbers as possible. That is, we want a number which has many *whole divisors*, or whole numbers which, when multiplied by one another, equal our base. Ideally, our base would be an *abundant number*, or a number for which the sum of its factors is greater than twice the number itself; that is, if one takes all of its factors and add them together, the result would be greater than twice the number. For example, the smallest abundant number is represented in base ten by 12, the number twelve; to prove this, simply add all of its factors together and determine whether the result is greater than twice twelve, or twenty-four. The divisors of twelve are 1, 2, 3, 4, 6, and 12; the sum of these divisors is 28. Twice twelve is 24; since 28 is greater than 24, we know that 12 is an abundant number.[Ɛ] An abundant number would be ideal because that means we have a large number of factors. Abundant numbers, written in base ten, include 12, 18, 20, 24, 30, 36, 40, 42, 48, 54, 56, and 60.

Why is it important to have a large number of divisors? Because it makes our mathematical calculations much easier. For example, assume for a moment that we were using base seven. Now assume that your dollar is made up of seven cents (logical in a base seven system), and attempt to divide your dollar in half. You cannot, unless you have a half-cent coin in your possession. Furthermore, writing $\frac{1}{2}$ in base seven is impossible with exact precision; one must resort to repeating digits to infinity, which is not only inaccurate but also extremely cumbersome. Other bases don't have such problems with $\frac{1}{2}$; in octal, for example, $\frac{1}{2}$ is written as 0.4, while in base

[Ɛ]Strictly speaking, an abundant number is one for which the equation $\sigma(n) > 2n$ is true, where $\sigma(n)$ is the sum of all possible divisors of n, including n itself. But in layman's terms, we've got it right, so there's no need to be fussy.

seven it must be written $0.\bar{3}$.[10] As another example, in base ten the common fraction $\frac{1}{3}$ must be written with a repeating decimal (also, coincidentally, $0.\bar{3}$), making this very commonly used fraction very difficult to work with in this base.

Fourth, we want to make sure that our base not only has as many factors as possible, but also has the *right* factors. Some fractions are used more than others; for example, while in daily life we frequently have need to divide a whole into thirds, we rarely have a need to divide a whole into sevenths or elevenths. We want to make sure that these common fractions—like halves, thirds, quarters, and so on—are easy to divide, which means that we want 2, 3, 4, and possibly others as even divisors. One result of this requirement is that we've already ruled out all odd numbers (that is, numbers which are not divisible by 2), which significantly reduces our pool of candidates.

Furthermore, these divisors and others stand out as uniquely important for another reason. Dividing into halves is a fundamental operation which will be performed more often than any other. Dividing into thirds and quarters will come closely after halves in frequency. Those thirds and quarters will themselves frequently be divided into halves, yielding sixths and eighths. Therefore, the divisors 2, 3, 4, 6, and 8 must be looked at carefully when examining possible bases. The base which makes these divisions easiest will show itself to be superior in a very important way.

Some numbers are also significant simply because of their nature as numbers. The number one is an easy example; however, all numbers have one as an even divisor, so we need not worry about that. 2, however, is only a divisor of even numbers, and it is an extremely significant number. It is the first even number, and it's the only even number that is also *prime*; that is, divisible only by itself and one. In geometry, we also require two points to draw a line segment, making it significant in that sense, as well. Three is another significant number; it produces the first polygon, the triangle, which is extremely important in mathematics and geometry and is the basis for trigonometry. Four is also very important; it is the first number that is not prime (it is divisible not only by itself and one, but also by two), and in geometry it is the first number of points which can produce a shape in three dimensions, a shape which we call a *tetrahedron* (in layman's terms,

[10]When a digit is repeated to infinity, it is common practice to simply write it once and put a line over it. The same goes for patterns; for example, 0.342342342342… could also be written $0.\overline{342}$.

this is basically a pyramid). Furthermore, it can also produce more than one shape; in three dimensions it produces a tetrahedron, while in two it produces a square, and is the lowest number capable of such a feat. To be truly useful, a base must include at least these basic and very important numbers as even divisors; a base which lacks them must be considered imperfect to that extent.

Finally, and to a much lesser extent, we ought to have some basis on our bodies for choosing our base. Base ten is a very common base in part because man has ten fingers on his hands. However, other bases can likewise be justified by our biology. We have two upper and two lower limbs, providing a justification for bases two or four; we have three knuckles on each finger, providing a basis for base three; on each hand, excluding thumbs (which can be used for counting the knuckles) we have twelve knuckles, providing a basis for base twelve[11] or for base twenty-four; and we have twenty fingers and toes together, providing a basis for base twenty. While lacking some biological basis is not a fatal characteristic, it is certainly a negative from the point of view of selecting a base.

At this point, we have determined the most salient criteria for determining how good or how bad a given base is. How does our current base measure up? Should we retain it as the best possible base? Or should we discard it in favor of a better one?

4.2 The Failures of Decimalism

We've identified five major considerations when determining how good or bad a given base is:

1. Small enough to have a reasonable number of symbols;
2. Large enough to ensure numbers of reasonable length;
3. An appropriately large number of whole divisors;
4. Having the most important numbers as whole divisors;

[11] In case this isn't clear: one can easily count in base twelve on one's fingers simply by using the thumb for counting the knuckles on the other fingers. In this way, one can count to twenty-four ("24" in base ten or "20" in base twelve) without any difficulty, allowing significantly more flexibility than simple finger-counting in base ten, which can only bring one to less than half of that precision. For further demonstration, refer to figure 3 on page 1↊.

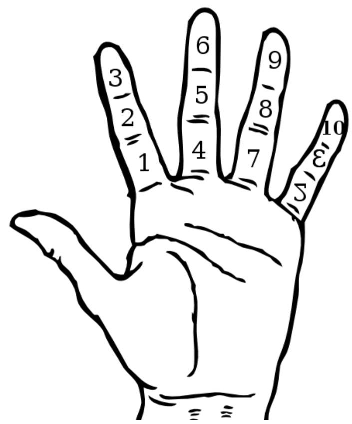

Figure 3: A diagram demonstrating an easy method of dozenal finger-counting.

5. Some biological basis for its selection.

Our current system of base ten (which I'll now refer to as "decimalism") fails on two of those five, giving it an overall grade of only 60%, which is failing (at least it was when I went to school). If no other base can get a better grade than this, then base ten will still remain the best base for us to choose; however, it still wouldn't be very good, and would be a necessary evil at best.

For the first two of these criteria, ten really isn't a bad base at all. Ten symbols is a reasonable number, not too great a tax on memory, and numbers of reasonable size are of reasonable length.

However, when we arrive at divisors, ten is quite literally a disaster. It's not a prime number, so it contains more than simply itself and one; however, it contains only two other divisors besides these, two and five. While two is, of course, a requirement of any good base, five is a useful one but not particularly necessary. Furthermore, the extremely useful divisors three and four are totally neglected by ten. This means that the extremely common unit of one-fourth ($\frac{1}{4}$) must be written with two digits to be accurate (0.25), while the extremely common unit of one-third ($\frac{1}{3}$) requires an infinite number of places to be written with accuracy ($0.\bar{3}$). Many other fractions are extremely difficult to write with ten; some of these, like three and four, are very important.

Divisors of Ten	
2	0.5
3	$0.\bar{3}$
4	0.25
5	0.2
6	$0.1\bar{6}$
7	$0.\overline{142857}$
8	0.125
9	$0.\bar{1}$

Table 4: Divisors of ten, written out in base ten place notation.

Table 4 on page 1Ɛ shows the values, written out in base ten, of all the single-digit fractions in base ten, except for zero and one, which are equally simple in all bases and therefore uninteresting here. Of those eight fractions, only half can be written out with accuracy in a finite number of digits. Of these, one requires three digits, one requires two, and two require one. Fully half of these fractions are *irrational numbers*, incapable of being written out without an infinite number of digits.

Truly, ten must be considered an abject failure where the number of divisors is concerned. We could have predicted this, since ten is not an abundant number (twelve is the smallest of all abundant numbers); however, it was useful for us to analyze the question in more detail than that prediction would permit. Furthermore, lacking three and four as divisors, ten must also be considered miserably inadequate concerning the number of the most important factors that it contains. Since it cannot deal comfortably with thirds and quarters, it cannot deal comfortably with half a third—a sixth—or half a quarter—an eighth—either. Ten as a base could be worse, of course; it does have some good characteristics. But it could also be much better. So we must, after all, seek out a better base, if we can find one.

4.3 The Glory of Dozens

There are, of course, lots of other possible bases. We'll begin with a few assumptions, however. The first is that any base smaller than eight is too small; it will yield numbers that are just too lengthy to deal with comfortably. The second is that any base higher than sixteen is too large; it will require too many symbols to use comfortably. These first two assumptions serve to eliminate the first two considerations we discussed above[12] for determining what makes a good base. This means that we can focus primarily upon the issue of divisors (the biological basis issue, while interesting, is not particularly dispositive.)

The third assumption is that, when working with bases larger than ten, we will use capital letters for those digits which are higher than ten but less than the base itself; so, for example, when working with hexadecimal, we will use numbers of the following type:

0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F, 10

You should recognize this convention from our demonstrations regarding hexadecimal numbers earlier.[13] Finally, the fourth and last is that odd numbers simply do not make acceptable bases because they lack the number two as an even divisor. These assumptions settled, we can proceed to consider the advisability of bases between eight and sixteen.

4.3.1 The Case for Dozenalism

To determine which of these bases would be the best to use, it's important that we compare them both on the basis of number of divisors simply, and on the number of important divisors. The best way to do this is to simply set them up beside each other and see which ones look better.

Table 5 on page 21 shows that most even numbers within our range have precisely the same number of divisors. For example, of the five even numbers shown, three have precisely four divisors; namely, eight, ten, and fourteen. One of these numbers has one additional divisor, five; namely, sixteen. The last, however, has an astounding six even divisors, and that is twelve. Once again, this is something which we might have predicted beforehand, since twelve is the only abundant number within our range; however, it is still helpful to examine the numbers individually, to see precisely how significant the difference really is.

So we've established that twelve is the superior base concerning the number of divisors. Let us now examine which base is superior in terms of number

[12] *See infra*, section 4.1, at 16.
[13] *See supra*, section 3.2, at 11.

Divisors of Various Bases	
8	1, 2, 4, 8
10	1, 2, 5, 10
12	1, 2, 3, 4, 6, 12
14	1, 2, 7, 14
16	1, 2, 4, 8, 16

Table 5: A comparison of divisors for even bases between eight and sixteen, written in base ten place notation.

of *important* divisors. This comparison will also include which base has the fewest irrational divisors, and how long each irrational number's repeating portion is. (It is, naturally, easier to deal with shorter repeating periods than longer ones.)

Table 6, on page 22, compares the fractions of each of these proposed bases, written out in terms of themselves (that is, in their own bases), side by side. I think you'll agree that this comparison is illuminating, even more so than that in table 5. Certain interesting patterns emerge, of course; one noticeable such pattern is that the fraction of the number immediately below the base number itself is always $0.\bar{1}$. But while interesting, this is relatively minor. Several quite important patterns are displayed on this table which will have a great bearing on which base we decide is superior.

One such pattern is that the fractions involving prime numbers, excluding two—in this table, 3, 5, 7, and 11—seem particularly troublesome. Three, for example, causes problems in all our bases except for twelve. Five is a mess in all our bases except for ten. Seven likewise causes problems in all bases except for fourteen. Eleven only appears as a fraction in three of the proposed bases; however, it results in an irrational fraction in all three. It is clear, then, that the prime numbers are significant problems for all our proposed bases. However, only one of these prime numbers, 3, was identified earlier as a particularly important divisor for a good base to have. That base which is best able to handle three, twelve (which handles three in a single digit), therefore has a significant advantage over the rest.

Another pattern is that when the half is an even rather than an odd number, the divisors of that half, including its own half, will also turn out to be quite easy to deal with. Table 6 displays this fact clearly. Those bases

	Eight	Ten	Twelve	Fourteen	Sixteen
2	0.4	0.5	0.6	0.7	0.8
3	$0.\overline{25}$	$0.\overline{3}$	0.4	$0.\overline{49}$	$0.\overline{5}$
4	0.2	0.25	0.3	0.37	0.4
5	$0.\overline{1463}$	0.2	$0.\overline{2497}$	$0.\overline{2B}$	$0.\overline{3}$
6	$0.1\overline{25}$	$0.1\overline{6}$	0.2	$0.2\overline{49}$	$0.2\overline{A}$
7	$0.\overline{1}$	$0.\overline{142857}$	$0.\overline{186A35}$	0.2	$0.\overline{249}$
8	0.1	0.125	0.16	0.1A7	0.2
9		$0.\overline{1}$	0.14	$0.\overline{17AC63}$	$0.\overline{1C7}$
10		0.1	$0.1\overline{2497}$	$0.1\overline{58}$	$0.1\overline{9}$
11			$0.\overline{1}$	$0.\overline{13B65}$	$0.\overline{1745D}$
12			0.1	$0.12\overline{49}$	$0.1\overline{5}$
13				$0.\overline{1}$	$0.\overline{13B}$
14				0.1	$0.1\overline{249}$
15					$0.\overline{1}$
16					0.1

Table 6: A comparison of fractions for even bases between eight and sixteen.

which have odd halves—in this table, ten and fourteen—do not benefit from the divisors of their halves. However, those which have even halves—eight, twelve, and sixteen—reap the rewards by having rational, and short, divisors of those halves. Let's look at sixteen and twelve as good examples of this. Base sixteen has a half of 8, the divisors of which are four and two. Two will be single-digit for all our bases, since they are all even, so let's take a look at four in base sixteen. It's a simple, rational number, written out with a single digit.

Even more impressive in this regard is twelve. Base twelve has a half of six; six has the divisors two and three. Again excluding two, which is an advantage for all even bases, this fact makes twelve the only one of our bases which has the extremely important third as a rational number—and that in a single digit! Truly, twelve stands out pretty clearly in this table as an extremely good base, at least as far as divisors are concerned.

Now, all of these bases have many irrational numbers in this table. Sixteen stands out in this regard, as nearly 70% of its fractions are irrational; in light of this, it fails dismally as a proposed base. Eight and fourteen come next, with fully half of their fractions irrational. This is clearly too many, and we must reject them as bases on this account. Ten is also a terrible failure, with approximately 40% of its fractions irrational. Twelve stands out as the clear winner, with only a little more than a third, four, of its eleven sub-base fractions being irrational.

None of these totals, of course, strike one as exceptionally good; however, it's clear that one is better than all the others. Furthermore, we must look not only at *how many* irrational fractions our proposed base has, but also at *which* of those fractions are irrational. In this regard, twelve once again emerges as a singularly excellent base.

In base ten, the extremely common third is a cumbersome irrational number. This is likewise the case with half of a third, or a sixth, which computes to a number even more cumbersome. Furthermore, accurately writing the also common fourth requires two digits, and half of a fourth, or an eighth, is likewise rational but requires three digits to write. In base twelve, on the other hand, both the third and the fourth are simple, perfectly rational, single digits (0.4 and 0.3 respectively). Half of a third, or a sixth, is likewise a simple, rational, single digit (0.2). Half a fourth, or an eighth, is a simple and rational 0.16, requiring one fewer digit to write accurately than the same fraction in base ten. Base twelve really shines by making common, useful fractions, many of which are also extremely important mathematically, very easy to work with.

To sum up: base twelve has the largest number of whole divisors, as well as the largest number of useful divisors. It alone among reasonable choices for a number base keeps the extremely important fraction of a third rational, and does so in a single digit. It also keeps the vitally important quarter down to a single digit, and it makes the halves of a third and a quarter—a sixth and an eighth respectively—extremely easy to work with. It also includes all the mathematically important lower numbers as whole divisors, and is tied for the fewest number of irrational fractions among all reasonable bases.

This examination yields an inescapable conclusion: the best possible base for a number system written out in place notation is twelve. Therefore, we will adopt twelve as such, and we shall immediately begin to reap the benefits of having selected such an excellent base. There are, however, a few other questions which still need to be addressed; these will be the subject of the next few sections.

4.3.2 Possible New Digits

Up until now, we have been utilizing a simple method for representing the additional symbols needed for bases higher than ten: we've simply used capital letters in alphabetical order. This is the pattern used by computers for hexadecimal; however, while it is good for computers, it isn't good for men. Numbers are fundamentally different from letters; they represent quantity, not sounds, and consequently look different from letters. Furthermore, the letters "A" and "B" (we only need two additional symbols for base twelve) simply don't blend aesthetically with the other numbers.

One suggestion has been more or less universally adopted; namely, the replacement of the "decimal point," which separates the whole from the fractional parts in a number which has both, with a "dozenal point." That dozenal point has been represented by the semicolon, ";". From now on, dozenal numbers can be easily identified whenever they contain that semicolon. The dozenal point performs the same function in the dozenal system as the decimal point does in the decimal; that is, it separates the whole part of the number from the fractional part. As an example, the number "three and a half" is written in dozenal as 3;6.

As for the two new digits, many different solutions have been tried. There are two main streams of dozenalism (that is, the system which uses base twelve, based on the dozen), the American and the English. Neither has officially endorsed any set of new digits; however, the general usage of the two streams is pretty well established, though individual opinions often vary.

The Dozenal Society of America (DSA) has advocated several different symbols. Originally, under the influence of F. Emerson Andrews (the father of modern dozenalism), the symbols X and E were almost universal. The X was a nod to the Roman numeral of the same shape, and the E was intended to evoke the the word "eleven." However, these symbols are bulky when used with other digits (consider 4X;E7, for example), and also fail to blend well with the other digits (which is true for almost all the attempts at new ones). For a time, the DSA used * and # (yes, really), because they appeared on the standard Bell Telephone phone, and therefore would be familiar to most Americans. These days, the DSA has returned to the digits they began with, very stylized versions of X and E; roughly, they look like χ and ϵ. All of these proposed digits, however, suffer from the same problems as plain X and E; namely, they just don't blend well with the other digits. A different set of digits would be preferable.

Our brethren across the pond, however, arrived at an excellent solution. It seems that Sir Isaac Pitman, inventor of Pitman shorthand, was also a dozenalist, and he devised some proposed characters for the two additional digits that the dozenal system requires. These two symbols are variations on those for 2 and 3; they have the shapes of numbers, and thus blend in quite well with the other digits which we already use. These symbols are ↊ and ↋.[14] Furthermore, these are the symbols utilized by Tom Pendlebury of the Dozenal Society of Great Britain, whose monumental work *TGM* is pivotal in the dozenal community and which we'll discuss at some length shortly.[15]

Now, many people have different opinions on these characters. Some support the other major opinions already discussed here; others advocate some variations on ↊ and ↋, similar symbols but not identical; some even argue for a completely new set of digits, including for 0–9. However, the most reasonable solution, which uses digits most likely to be familiar to learners, and which blend best with the digits we already have, yet which presents only a slight learning curve to prospective dozenalists, is the solution that we have proposed here. A simple number line, displaying the characters in

[14] The symbols you see in this document—namely, ↊ and ↋—were created by the author in Donald E. Knuth's Metafont font design program. The author is not a type designer; while he believes he's arrived at decent characters, any and all improvements would be most appreciated. See the dozenal package at http://www.ctan.org for more information.

[15] *See supra*, section 5.2, at 38.

counting order, goes as follows:

1 2 3 4 5 6 7 8 9 ᘔ Ɛ 10 11...

There are some objections to these symbols; however, the most significant one appears to be the difficulty of representing them in seven-segment displays. Seven-segment displays are, of course, quite common in devices such as calculators, alarm clocks, gas pumps, and the like. The argument against the Pitman characters on this ground is that, since these displays are quite widespread, it would be preferable to choose characters which can be easily represented on such displays. However, for the reasons below, this argument is insufficient to undercut the venerable Pitman characters.

In the first place, it is perfectly possible, and indeed easy, to represent these symbols in a seven-segment display. Some argue that the only possible representation of ᘔ in seven segments is indistinguishable from two. However, there are a number of possible realizations for ᘔ in seven segments, many of which are not ambiguous. One such suggestion, which the author deems quite aesthetic, is displayed in figure 7 on page 27. Some further suggest that the confusion between the representation of Ɛ and that of the letter "E" renders Ɛ undesirable. However, "E" rarely appears on seven-segment displays; occasionally it appears on simple calculators indicating an error, but that function could just as easily be fulfilled by some other character. All in all, the limitations of the seven-segment display do not really affect the suitability of ᘔ and Ɛ.

Furthermore, seven-segment displays are antiquated already and are becoming more so by the day. Even common microwaves tend to have significantly more versatile displays in this day and age. Over time, we will see fewer and fewer such displays, until in the fairly near future they will probably no longer exist, at least in any significant numbers. Of course, they exist *now*, and dozenalists need to take that into account. However, sufficient account of these older displays has already been taken simply by devising acceptable representations for the new numerals on such displays. It's certainly important to account for older technology; however, such older technology presenting such minimal difficulties should not stand in the way of an otherwise appropriate solution.

Is it possible that better symbols could be determined? Perhaps. These symbols may not be perfect, and many will be unhappy at this author's advocacy for them. However, they are well-known within the dozenal community, well-established, easily produced, and well integrated with current decimal

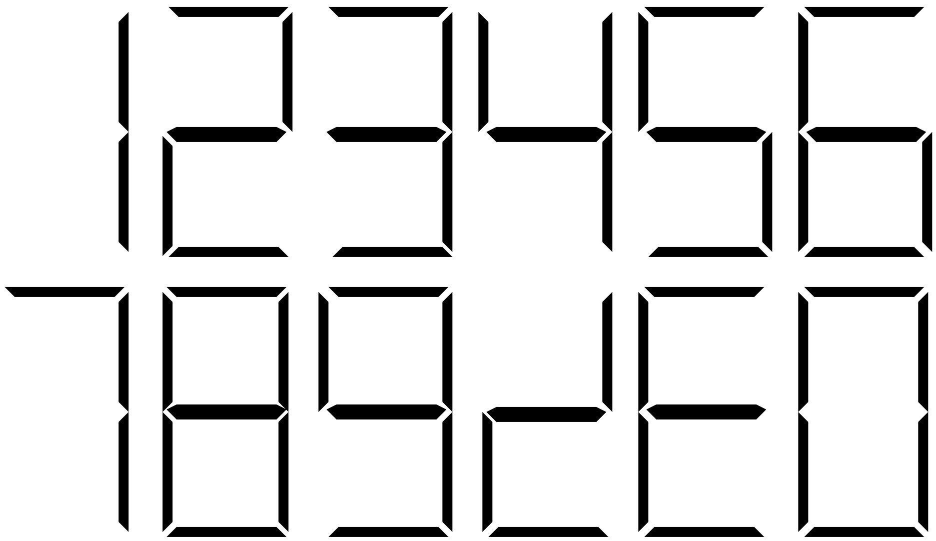

Figure 7: A figure showing simple seven-segment displays for all numerals, including the Pitman characters ↊ and ↋.

digits. As such, they blend well with current digits, are easily recognized as numbers, and present the least amount of difficulty to beginning dozenalists.

Dozenalism requires unity in the question of additional digits. Without consistently presenting the new digits, it will always be difficult to convince others of the viability, and indeed the superiority, of the dozenal system. Those symbols most likely to gain such universal acceptance are those with the largest body of already existing work; recognized by the largest body of people; most easily drawn and produced by those knowing our current number system; and well-integrated with the current digits. The Pitman characters are the answer to this need. Therefore, though they may not be perfect, they are what dozenalism requires.

It is this author's hope that, given their already widespread use and their use in the pivotal work of Pendlebury, the dozenal community will rally around these digits, and the dozenal movement can progress with a consensus concerning its extra figures. With suitable digits and an agreement to use them, dozenalism will be a mathematical force to be reckoned with.

4.3.3 The Need for Better Words

As mentioned above,[16] even though we've found a base that is far superior to our current system, we still need to find a new way to talk about numbers. While talking about them is certainly manageable with our current language, anyone who works frequently with numbers—that is, almost everyone—would quickly find our current system cumbersome, particularly when dealing with numbers greater than 1000 (decimally, 1728).

Traditionally, of course, the number "twelve" has been referred to as a dozen; indeed, this is precisely where the name "dozenalism" comes from. A dozen dozen, written in dozenal as 100 and in decimal as 144, was known as a "gross"; a dozen gross, 1000, was traditionally known as a "great-gross." Beyond that, however, there are no more specialized terms. We could speak of a dozen great-gross, or a gross great-gross, or three dozen gross great-grosses, and still manage to reap many of the fantastic benefits of the dozenal system; however, as mentioned above, we'd quickly find it cumbersome to speak about the calculations that our number system has made so much more efficient.

In the decimal system, on the other hand, we have a variety of different terms. This is not to say that our current system makes much sense; it doesn't. Furthermore, it varies considerably between British and American usage. For example, in American English a "billion" is a thousand times a million, written out 1,000,000,000. However, in British English, this is a "milliard," and a billion is rather a *million* times a million, or 1,000,000,000,000. However, there is still a standard way of referring to numbers, and that standard is reasonably short and easy to use. No such system exists for dozenal, at least in the common parlance.

However, Tom Pendlebury of the Dozenal Society of Great Britain has devised an excellent system which takes all the guesswork out of counting in dozenal. He provides a quick, easy way to say any given number, as well as a rational and standard way of handling powers of a dozen (the way that "thousand," "million," "billion," and so on in decimal simply do not). While many dozenalists have invented such a system, Mr. Pendlebury's is by far the most cohesive and rational, and because of its association with TGM,[17] it provides the best chance of presenting a single and coherent system to a decimal world.

[16] *See supra*, at 15.

[17] *See infra*, Section 5.2, at 38.

Basic Dozenal Counting			
1	one	11	onezen one
2	two	12	onezen two
3	three	13	onezen three
4	four	14	onezen four
5	five	15	onezen five
6	six	16	onezen six
7	seven	17	onezen seven
8	eight	18	onezen eight
9	nine	19	onezen nine
Ꮭ	ten	1Ꮭ	onezen ten
Ɛ	elv	1Ɛ	onezen elv
10	zen; onezen	20	twozen

Table 8: Basic Pendlebury counting in dozenal, from one to twozen.

As can be seen in table 8 on page 29, this system has a number of admirable qualities. First, it changes as little as possible of the already existing numbers; one through ten are totally unchanged, and eleven is shortened to elv simply for the convenience of having a single syllable for higher numbers such as elvzen. The word for "twelve" is clearly derived from "dozen," but is more versatile than either "twelve" or "dozen" for use in compounds, such as "fourzen seven" (a clear winner over competitors "four dozen and seven" and "fourtwelve and seven").

Furthermore, our current decimal system of counting contains an odd irregularity: the teens. Most numbers are counted by putting on a simple prefix for the number of tens, and then stating the number of ones. Forty-seven, eighty-six, seventy-four—all of these work the same way. However, the numbers higher than ten and less twenty are a notable and confusing exception. First, eleven and twelve follow no pattern whatsoever, being totally unanalyzable roots. Second, thirteen through nineteen reverse the pattern, instead using a *suffix* when all eighty other comparable numbers use a *prefix* as described above. The Pendlebury dozenal system solves that irregularity and solves it admirably, using still easily pronounced number names arranged in a systematic way.

Table 8 (on page 29) makes it easy to count all the numbers from one (1)

Full Pendlebury System				
Prefix	Number	Zeroes	Exp.	Decimal
Zen	10	1	10^1	12
Duna	100	2	10^2	144
Trina	1,000	3	10^3	1,728
Quedra	10,000	4	10^4	20,736
Quen	100,000	5	10^5	248,832
Hes	1,000,000	6	10^6	2,985,984
Sev	10,000,000	7	10^7	25,831,808
Ak	100,000,000	8	10^8	429,981,696
Neen	1,000,000,000	9	10^9	5,159,780,352
Dex	10,000,000,000	Ɛ̷	$10^{Ɛ̷}$	61,917,364,224
Lef	100,000,000,000	Ɛ	$10^{Ɛ}$	743,008,370,688
Zennil	1,000,000,000,000	10	10^{10}	8,916,100,448,256
Zenzen	10,000,000,000,000	11	10^{11}	106,993,205,379,072
Zenduna	100,000,000,000,000	12	10^{12}	1,283,918,464,548,864
...				
Dunduna		22	10^{22}	1.144754599... x 10^{28}

Table 9: The full Pendlebury system of dozenal counting.

to elvzen elv (ƐƐ). However, when we move on to a full gross (100), we still seem to have the same problem of running out of words. But Pendlebury again presents an excellent solution, which can be found in its entirety in table 9 on page 2Ɛ̷.

Notice how intensely rational this system is. The name of each power of a dozen is based on the number of zeroes that must be added to it. The first is named "zen," as already examined; this word is derived from "dozen," and allows not only for a reasonable word on its own but for easy compounding with other words, such as "threezen seven" for 37 and "sixzen elv" for 6Ɛ. Otherwise, however, all the names are clearly related to the number of zeroes that must be placed after the one for the simple power of zen; this is an immense improvement over the current decimal system of naming, as is demonstrated in table Ɛ̷ on page 2Ɛ.

For example, we can see clearly that "duna" evokes the notion of "two,"

Pendlebury and Decimal Compared			
Power	Pendlebury	Current	Number
1	Zen	Ten	10
2	Duna	Hundred	100
3	Trina	Thousand	1,000
4	Quedra	Ten Thousand	10,000
5	Quen	Hundred Thousand	100,000
6	Hes	Million	1,000,000
7	Sev	Ten Million	10,000,000
8	Ak	Hundred Million	100,000,000
9	Neen	Billion	1,000,000,000
↊	Dex	Ten Billion	10,000,000,000
↋	Lef	Hundred Billion	100,000,000,000
10	Zennil	Trillion	1,000,000,000,000
11	Zenzen	Ten Trillion	10,000,000,000,000
12	Zenduna	Hundred Trillion	100,000,000,000,000
13	Zentrina	Quadrillion	1,000,000,000,000,000

Table ↊: Comparing the Pendlebury system of powers with the American decimal one.

while being different enough to make it clear that it's not referring simply to the number two. Similarly, "trina" evokes three, "quedra" four, and so on, each word being derived from well-known Latin or Greek roots which should be familiar to most English speakers, and will be easily learned by others. "Lef" is clearly related to "eleven," while "zennil" is related to the dozen word "zen," while clearly showing that we're referring here to a dozen zeroes and not merely to one. After that, one simply adds "zen" to the appropriate prefix to continue getting the larger and larger numbers.

In our current system, on the other hand, the words denoting the various powers of ten are not obviously related, if they're related at all, to the number of digits the number will contain. Table ↊ on page 2↋ makes this reality quite clear. While the words "billion" might evoke the meaning of "two," beginning as it does with the "bi" of "bicycle," "biennial," and any number of other words, in reality it has nothing to do with two; a "billion" means that the number has nine zeroes. "Ten billion," on the other hand, means that one has ten zeroes, not that one has ten plus two zeroes, as the name might imply. A "trillion" has nothing to do with having three zeroes in the number, nor does it have anything to do with being three times the last number, the billion; rather, it simply means that one's number has a dozen zeroes at the end. Similarly again with the next step up, the quadrillion. The British system varies somewhat from this, where what Americans call a billion would be called a "milliard," but it does nothing to solve this fundamental problem. Clearly, the Pendlebury system for dozenal counting is vastly superior to the current system of decimal counting. We will use it from this point onward.

That takes care of all numbers which are *greater* than zero; how do we handle those numbers which are *less* than zero? In the decimal system, these are called simply "decimals," which are essentially fractions written on a single line. Fractional parts, called in our current system "decimals," work exactly like the whole parts, except backwards. For example, 0.4 is, in base ten, often read "zero point four"; it is often, however, also read by its value, which is four tenths ($\frac{4}{10}$, or $\frac{2}{5}$). It is, essentially, $4\text{x}10^{-1}$. In other words, each place to the right of the "decimal point" indicates another negative power of ten. The first place indicates tenths; the second hundredths; the third thousandths; the fourth ten thousandths; and so on. When we see a complex number like 0.4367, we can not only read it in the common way, as "zero point four three six seven," but also by saying "four thousand, three hundred and sixty-seven ten thousandths," or $4{,}367\text{x}10^{-4}$. One can see, therefore, that the question of fractional parts and the question of negative powers of the base is the same.

Dozenal numeration provides the same sort of number, of course, utilizing the semicolon as the "dozenal point." To facilitate differentiating between base ten and base twelve numbers, we shall follow Pendlebury once again and refer to these fractional parts as "zenimals," and the semicolon as the "zenimal point." Just as the decimal point is pronounced "point" in common parlance, the zenimal point is pronounced "dit." For zenimals, the first place to the right of the zenimal point means twelfths; the second, grossths; and so on. So the zenimal 0;4, normally read "zero dit four," could just as easily be referred to as "four twelfths" (or its equivalent, $\frac{1}{3}$). In this way, it is precisely as easy to work with as our current decimal system.

However, Pendlebury's excellent system provides a still better way to refer to such numbers. We have seen that "zen" (or "zena") refers to 10^1, expressed in the decimal system as 12. Rather than make us bother with a whole new system for 10^{-1}, Pendlebury's system indicates simply affixing an "i" to the positive power name and resting at that. So 0;4 ("zero dit four") is also easily expressible as four *zenis*, the same way that 40 is easily expressible as four *zena* (though this will normally be abbreviated to "fourzen"). Similarly, 0;04 is four *dunis*, 0;004 is four *trinis*, and so on. When presented with a more complex zenimal like 0;579, the same principles allows one to refer to this as five duna sevenzen and nine *trinis*. It is, as always, a simple matter of counting the powers of zen; in this case, the number of digits to the right of the zenimal point. 0;0008 is not eight ten thousandths; it's eight quedris, just as 80,000 is eight quedras. The dozenal value of π, 3;18, is simply three and onezen eight dunis. It is a system quite admirable in its simplicity.

This also greatly facilitates speaking about multiplication and division by a power of the base. As an example, let's take the current year in dozenal, 11Ɛ5 (which is "2009" in decimal). This would be pronounced, in the Pendlebury system, "one trina, one duna, elvzen and five"; however, doubtlessly a shorthand would arise similar to the one used in the last century for decimal years, and this would often be pronounced simply "onezen one elvzen five."[18] To *multiply* this by one trina, or 1,000, we simply move the implied dozenal mark at the end of the number three places to the right, just as if we were moving a decimal point. So 11Ɛ5 multiplied by a trina, 1000, equals 1,1Ɛ5,000.

We could just as easily, however, decide to multiple 11Ɛ5 by a trin*i*, which is precisely the opposite. That is, instead of multiplying by 10^3, we multiply by 10^{-3}. In that case, we just move the value three points to the *left* instead of the right, giving us 1;1Ɛ5. All the powers of zen can be treated thus; so, for example, a quedr*a* equals 10^4, or moving the dit four places to the right, while a quedr*i* equals 10^{-4}, or moving the dit four places to the left. There is no need to refer to multiplying by one hundred-thousandth; one is simply multiplying by one queni.

This eminently simple and logical system will greatly facilitate mathematics of all types.

4.3.4 Some Applications of Dozenal Numeration

We've established, above,[19] that the dozen is the best base *in the abstract*; but what particular applications of dozenalism would really make our math-

[18] This is similar, in other words, to pronouncing "1995" as "nineteen ninety-five" rather than "one thousand, nine hundred, and ninety-five."

[19] *See supra*, Section 4.3.1, at 20.

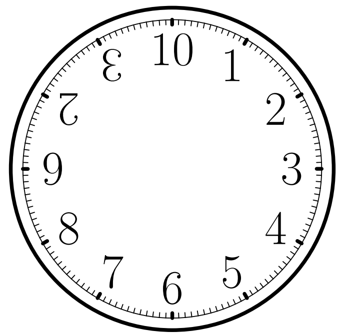

Figure Ɛ: A figure displaying the clock in dozenal numeration.

ematical lives easier and more effective? How long before we begin seeing real effects from its adoption?

Almost immediately; even waking up in the morning would be a new and easier experience. Figure Ɛ on page 32 displays clearly the advantages in time measurement that the dozenal system would very quickly bring. The clock is divided into twelve units (corresponding to five minutes), and each of those units are divided into twelve smaller units (each of which lasts a little less than thirty seconds). The advantages of the dozenal system for time are clearly illustrated.

No longer is there a need for strange machinations, by which the twelfth of the hour must be multiplied by five in order to come to the number of minutes. Rather, the twelve of the hour is the first digit to the right of the dozenal point. What we would now, in the decimal system, write as "4:30" would be written instead as "4;6"; the shorter hand points to the hour, and the longer hand points to the twelfth of the hour (the zeniHour), and so the number at which it points can be added to the time with no ambiguity. A quarter past the hour is equal to a quarter, 0;3; so a quarter past five is 5;3, while a quarter until six is 5;9, the larger hand of the clock pointing to the 3 and the 9, respectively.

Furthermore, so-called "military time," more properly called "twenty-four hour time" (or, in the dozenal system, simply two dozen hour time) would cease to require any mental mathematics whatsoever. Presently, one must take the number of hours after noon and add that to the decimal number 12; therefore, 19:00 in decimal refers to 7:00 *post meridiem*, or p.m., despite the fact that the number contains no digit 7. In the dozenal system, on the other hand, making a time p.m. is a simple matter of adding a 1 to the left of it. So 7;00 in dozenal is 7:00 a.m.; 17;00 in dozenal is 7:00 p.m. The whole system is much more intuitive and useful than our current decimal system, which we mangle in order to apply to a dozenal system of timekeeping.

This furthermore makes shift management and similar tasks immeasureably easier. Currently, a factory running three equal shifts throughout the day must begin and end them at odd, uneven hours; the first shift, for example, may run from 7:00 a.m. until 3:00 p.m., the second shift until 11:00 p.m., and finally the third back to 7:00 a.m. The dozenal system makes a three-shift day make sense. If the first shift begins at 8;00, then the second would begin at 14;00, and the third at 00:00 (or 20;00, whichever way of writing it is preferred). In the dozenal system, this is simple addition:

$$8 + 8 = 14 \tag{8}$$

$$14 + 8 = 20 \tag{9}$$

There is no need to add five to seven in order to get twelve, then go back to zero and add three more in order to arrive at 3:00 p.m. for the eight-hour shift. There is no need for a.m. or p.m. at all, because going from morning to afternoon is a simple matter of adding a one to the left of the number.

All in all, our system of measuring time is much better given to dozenal than to decimal numeration, and switching to dozenal numeration consistently will make our management of that time much easier.

Another boon to human endeavor will come from the application of dozenal numeration to the study of statistics, and any other study which requires the use of percentages. Percentages, of course, are simple expressions of parts per hundred; when we say that 33% of the population voted for such-and-such, for example, we are saying that out of every hundred voters, 33 of them voted for the proposition. This system is logical when the inferior base of ten is being used; for the dozenal system, however, it clearly will not do.

Rather than percentages, the dozenal system offers pergrosses; that is, parts per gross. All the many advantages of the easily dividable base of twelve will immediately arise to greet the ecstatic statistician. If he wants to say that two-thirds of the population preferred his particular candidate, there is no need for rounding or repeating fractions; he simply says that 0;4% of the population preferred it. Comparison of results will be much easier, as he can easily compare the actual numbers to all the even, clean, and rational

Dozenal Multiplication Tables											
1	2	3	4	5	6	7	8	9	↊	↋	10
2	4	6	8	↊	10	12	14	16	17	1↊	20
3	6	9	10	13	16	19	20	23	26	29	30
4	8	10	14	18	20	24	28	30	34	38	40
5	↊	13	18	21	26	2↋	34	39	42	47	50
6	10	16	20	26	30	36	40	46	50	56	60
7	12	19	24	2↋	36	41	48	53	5↊	65	70
8	14	20	28	34	40	48	54	60	68	74	80
9	16	23	30	39	46	53	60	69	76	83	90
↊	18	26	34	42	50	5↊	68	76	84	92	↊0
↋	1↊	29	38	47	56	65	74	83	92	↊1	↋0
10	20	30	40	50	60	70	80	90	↊0	↋0	100

Table 10: A table depicting dozenal multiplication tables from 1 to 10.

divisors that inhere in the base of twelve but not that of ten in a blink of an eye. What rounding he does have to do will have more probability of being cleanly rounded to a clean, even fraction. And finally, the greater divisibility of dozenal 100 (the gross) as compared to decimal 100 (the hundred) will permit much easier calculation of pergrosses than percentages in the first place, as the ratios will benefit from twelve divisibility in the form of easier reducibility. Furthermore, no change even in his writing will be necessary; since both one gross in the dozenal system and one hundred in the decimal system are written 100, the symbol "%" will continue to serve perfectly well to express pergrosses.

Multiplication tables will also be made easier, as a greater number of easy patterns for easy multiples appears. As in the decimal multiplication tables, the two columns and the base column are easy. Also, the column for half the base (decimal 5, dozenal 6) is also quite simple. In addition to this, however, the threes, fours, and eights follow definite and simple patterns. For the three column, for example, the multiples always follow a pattern of ending in 3, 6, 9, and 0, in that order. Similarly for the four column, they end in 4, 8, and 0, in that order. And again, in the eight column, the answers end in 8, 4, and 0, in that order. The nines are similarly simple, ending in 9, 6, 3, and 0, in that order. All in all, a full half of the columns in the dozenal tables follow a simple and set pattern.

Compare this to the decimal multiplication tables. The three column follows no discernible pattern; while the four column does, it's a five-number pattern, only repeated twice in the entirety of the table (4, 8, 2, 6, 0). Six is the same way (following 6, 2, 8, 4, 0), as is eight (8, 6, 4, 2, 0). Nine follows a pattern, but it is a full ten digits long, from nine to zero, meaning it only repeats once in the entirety of the table. (In the dozenal table, Ɛ has the same advantage.) The dozenal multiplication tables not only extend to higher numbers, but are also simpler, containing more patterns and thus presenting a shallower learning curve.

Prime numbers will also be much easier, for all prime numbers in dozenal will end in 1, 5, 7, or Ɛ.[1ᘔ] This means that all prime numbers greater than 3 are contained in the set $(6n \pm 1)$; in other words, the position of *all* prime numbers higher than 3 is governed by the number 6, a very important number in the dozenal base. It further means that a prime number must end in a prime number; notice that the 1, 5, 7, and Ɛ we mentioned earlier are all primes. In the decimal system, on the other hand, primes can not only end in 1, 3, and 7, they can also end in 9, which is *not* a prime number. This not only appears less orderly to the human mind; it also makes identifying a prime number more difficult.

By this point, we have examined different systems of numbers and different number bases for place notation; we have selected the best possible base of all reasonable bases; we have determined what symbols can best be used to represent our numbers; and we have determined what words can best be used to express numbers in our chosen base. The system we have arrived at is clear, logical, and mathematically sound. However, there still remain objections to the system as it stands. Let us proceed to those objections, answering them in turn, and see if any are sufficient to prevent the adoption of the excellent number system we've examined up to now in these pages.

[1ᘔ] In mathematical terms, all prime numbers are members of the set $2, 3, (6n \pm 1)n \in \mathbb{N}$. Don Hammond has shown this in his brief *Base Twelve and the Prime Numbers*, http://-www.dozenalsociety.org.uk/leafletsetc/sixesprimes.html, whence this minimal prime set was taken.

5 Objections to Dozenalism

Difficult as it might be for a committed dozenalist like the present author to imagine it, some are still not satisfied that a common switch to the dozenal system would be appropriate. Some of the most common objections have been collected here.

5.1 The Cost of Conversion

Possibly the most common, and also the most basic, objection to public adoption of the dozenal system is that the effort simply isn't worth it. Mathematics textbooks would need to be reprinted; people would need to be familiarized with the new numbers; everything, in effect, would need to be redone. Why bother? When most people have never even noticed that the current system is base ten, much less that there might be a better base, why should we dedicate such an effort to it, particularly when that effort would probably be greater than we can bear?

Ultimately, the answer to this objection is always the same: if something is better, then we ought to pursue it. We have amply shown, even in this short document, that the dozenal system is simply better. So we ought to pursue its establishment.

More particularly, however, several responses can be made.

First, we certainly would need to reprint mathematics textbooks, and basically all books, so that their page numbers would conform to the new system if nothing else. However, this would not be prohibitively expensive, or in fact expensive at all, because it need not be done all at once. Indeed, in many cases it need not be done at all.

For textbooks, for example, the easiest way to transition to the new system is simply to print all new textbooks with the dozenal standard, and allow the old textbooks to retain their older, inferior decimal numbering. The same would be true of all books. This author, for example, has a fairly extensive library, nearly all of which is printed assuming a decimal numbering system; why should he be required to purchase all new copies of these books? Better to leave them as they are; the new books I purchase will have dozenal measuring, and eventually the decimal books will be phased out, continuing as collectors' items and hobbyists' trophies.

Would it be difficult for people to know and be proficient in two number systems simultaneously, while the transition is taking place? Absolutely not.

Similar transitions, of a similarly or even more greatly extensive variety, have been successfully accomplished many times and in many countries. An excellent example is the recent metrication in Canada. Canadians were, in the not distant past, just as wedded to feet, miles, gallons, and pints as we here in America are. However, it was decided (erroneously) that metrication was the appropriate course, and people began to think in both the imperial and the metric systems, and gradually the imperial system was phased out, resulting in the thoroughly metricized state that currently resides on our northern border. We think in our measurements nearly as much as we think in our numbers; why should not the same sort of process work for dozenalization?

Similarly, in Great Britain people had been used to the daily use of a tripartite, multi-base system of currency, known commonly as "pounds-shillings-pence," or £.s.d. In this interesting (though not to say ideal) system, there were twelve pence to a shilling and twenty shillings to a pound. There were also a myriad other divisions of currency in common use, including guineas, crowns, florins, and farthings, among others. The British people were quite used to this system, and many were quite fond of it, particularly of the enormous number of divisors that the 180 (240 in decimal) pence permitted them. However, in 1180 (1968), decimalization of the currency was declared; in the future, there would be one hundred pence to a pound. Yet the British people managed, despite having to make such an enormous change in their everyday lives.

Perhaps an even more wide-ranging alteration was the Turkish transition from using the Arabic alphabet to using a version of the Roman. This occurred in 1148 (1928). For centuries, Turks had written their language in the Arabic script. All their books were printed in Arabic; they had only written Arabic; they were all thoroughly used to Arabic. However, it was determined that a modified Roman alphabet would be a better fit for the Turkist language. Therefore, the change was introduced. Books from then on were printed in the new alphabet; children who had been taught Arabic transitioned into learning Roman; adults learned the new alphabet, as well. For a time, the two alphabets were in daily use, as people read their old Arabic books and then turned to their newer Roman ones. Yet they did it, because they determined that it would be better than what they had before.

Can we honestly say that a transition to dozenal would be any more difficult than this? Let us not seek what is easy; let us rather seek what is good. For number systems, dozenal is good. So let us pursue it.

5.2 The Metric System

A still more common objection is the metric system. If we based all our numbers on twelve, the argument goes, we would lose the enormous benefits that the metric system has brought to the whole world, except for the United States, which still muddles along quite happily without it. While it is true that the dozenal system and the current metric system would not get along, the dozenalist's response to this is essentially an elaborate "no great loss." Let us examine some of the reasons for this riposte.

5.2.1 The Faults of the Metric System

The metric system itself is inherently faulty. Not only are its most basic measurements poorly chosen, but those poorly chosen measurements are sometimes completely wrong. Furthermore, it ignores key natural quantities, making them extremely difficult to work with. Finally, the basic quantities of the metric system are faulty; they do not correspond to normal human experience, requiring the invention of ad hoc customary measurements for purposes of dealing with the real world, such as the "metric foot" and the "metric pound."

The metric system is, supposedly, based entirely on the meter. (In reality, it is based upon several different measurements, including the meter. For example, the gram is one thousandth of a cubic decimeter.) The meter itself is supposed to be a universal measurement, dependent upon no particular nation, independently verifiable in its length, and scientifically based. The French revolutionaries who devised this system believed that one fraction of the circumference of the earth was an appropriate way to choose this length. Their "meter" was one ten thousandth of the distance from the equator to the north pole.

Or they thought it was, anyway. In reality, they had an inaccurate measure of the distance from the equator to the pole, and thus the meter is no such thing. Not only did they measure the distance from London to Barcelona, and extrapolate to the total distance from that, but they also assumed that the earth is a perfect sphere, which it isn't. However, by the time anyone realized this the meter was permanently enshrined by a platinum bar sitting somewhere in Paris, and the virtues of this platinum bar as an "objective" standard were being trumpeted throughout the world. So metricists, desperate to uphold the standardization of said platinum bar, finally determined a new way to define the meter, which would keep it the size that it had always been (that is, not one ten thousandth of the distance from the equator to the pole, but rather the same length as this platinum bar in Paris): they measured that platinum bar with a laser and defined the meter based on the speed of light. In decimal numeration, the meter is defined as the distance travelled by light in $\frac{1}{299792458}$ seconds. Yet this totally arbitrary measurement is still trumpeted as the basis for a rational and scientific system of measurement.

Furthermore, many of the standard measurements that people claim to be metric are absolutely not, but simply metricizations of older, standard measurements. An example is 35mm film, which is really $1\frac{3}{8}$ inch film.[1Ɛ] The real measurement of the film is not, in fact, 35mm, but 34.975mm. Yet the metricists will routinely take credit for this extremely versatile and well-sized standard film, despite the fact that it was standardized on imperial units and only approximately and clumsily transferred to metric ones.

Furthermore, as a "scientific" base for a system of measures, the metric system truly fails miserably. Take the most fundamental physical quantity of life on earth, the acceleration due to gravity. The metric system makes this an awkward $9.80665 m/s^2$, usually abbreviated to 9.81. Granted, in imperial units the quantity is equally awkward, equalling $32.175 ft/s^2$; however, that fact does not make the metric quantity any more useful. The metric system further makes absolutely no provision for time whatsoever; it neither makes time conform to its otherwise decimal divisions, nor does it attempt to define a standard unit of time in terms of its decimal divisions. Time, rather, continues to be mixed-base, with 20 (twozen) hours in a day divided into 10 (zen) units of five minutes each, for a total of sixty minutes in an hour. The metric system's neglect of time, through the lens of which men by necessity view everything, is an enormous failure considering its claims to being a modern and scientific system.

Even so, the supposed superiority of the metric system is belied even by those who use it. It is common practice in metric countries to refer to weights in so-called "metric pounds," which are half a kilogram, as well as to use "metric pints," equal to half a liter, for determining appropriate drinking quantities. Not to mention that boards are sold in standard lengths of 120cm, not in simple meters, because dividing a meter-long board into

[1Ɛ] *See, e.g.*, Joe McGloin, *Half-Frame Cameras*, 11Ɛ1, available at http://www.subclub.-org/shop/halfframe.htm. Last accessed 20 May 11Ɛ5.

thirds is unreasonably difficult given its base in ten.[20] And the thickness of the wood is measured as a "thumb," or 2.4cm![21] If the very people who use the metric system are forced to devise such new units on a regular basis, perhaps that speaks for the metric system's value for actual, daily use.

So once again, the dozenalist must respond to this objection with words to the effect of "no great loss." In fact, however, losing the metric system could well be an enormous *gain*. The ingenuity of dozenalists has produced an excellent, and much better, metric system in the dozenal base. That system will, briefly, by the subject of our next section.

5.2.2 TGM: An Improved, Dozenal Metric System

Tom Pendlebury, to whom we have referred frequently, has bequeathed a great gift to the mathematical world in his dozenal metric system, TGM. Standing for Tim, Grafut, and Maz, the three most fundamental units of the system, TGM presents, in the author's own words, "[a] coherent dozenal metrology based on Time, Gravity[,] and Mass."[22] Not only is the system *scientific*, in the sense that it provides useful units which can be easily converted in all the fields of science, but it is also *practical*, in the sense that the units it produces are usefully sized for everyday practical use. This is not the place for a full exposition of the system; for that, we refer the reader to the excellent booklet *TGM* produced by Mr. Pendlebury for the Dozenal Society of Great Britian,[23] which is freely available to all. However, a brief exposition of its most pertinent and commonly-used parts will doubtlessly be helpful.

As mentioned earlier, the mean solar day (in layman's terms, the average length of the day on earth) is already divided into dozenal parts; that is, it is divided into twozen (20) hours. Pendlebury sees no need to change this. A zeniHour (0;1 hours) is equal to five of our current minutes. Further dividing the hour into duniHours, triniHours, and quedriHours, Pendlebury

[20] Joan Pontius, *Metric Land, or: What I think of the metric system*, available at http://web.archive.org/web/20011102140224/www.rci.rutgets.edu/~jup/metric/metric_land.html. Last accessed on 21 May 11Ɛ5.

[21] *Id.*

[22] Tom Pendlebury, *TGM: A coherent dozenal metrology based on Time, Gravity and Mass* (The Dozenal Society of Great Britain), available at http://www.dozenalsociety.-org.uk.

[23] *Id.*

determined that the quedriHour produced the most suitable units for a system of measure. This quedriHour (0;0001 hours), equal to 0;21 seconds, is the most fundamental unit of TGM, and is called the Tim. Once we have the Tim, there is no need to utilize seconds anymore; an hour is simply defined as one quedraTim (10,000 Tim), and a full day, which is twozen quedraTim, is defined as two quenaTim (200,000 Tim). The beauty of this system is that it provides a sensible, scientifically derived fundamental unit of time, but still avoids upsetting our customary units of time. The common period of five minutes is certainly equal to one trinaTim (1,000 Tim), but it's also simply a zeniHour, or a twelfth of an hour, and can be written as simply as 0;1 hours. No TGM unit corresponds to a minute, but a duniHour is equal to almost half a minute (25 of the old seconds, in dozenal), and can be written as simply as 0;01 hours. The triniHour is only about twice as long as our second (it equals 2;1 second), and is written simply 0;001 hours. These units are conveniently sized, ordered dozenally, and leave our most important units unchanged.

It further forms the basis for the fundamental unit of length, the Gravity Foot, or Grafut. Man experiences one thing constantly, and that is the accleration due to gravity. This acceleration pulls us downward, in the old measurements, at about $32.1741 ft/s^2$ (or, as mentioned earlier, $9.80665 m/s^2$). However, if one uses Tims rather than seconds, one finds that the acceleration due to gravity is about $11\frac{5}{8}$ inches, or about 30 centimeters, per Tim per Tim. In other words, it is very close to the foot, a measurement which, give or take a little, was commonly used as a measure of length throughout most Western countries prior to metrification. Because this fact of gravitational acceleration is fundamental to life on earth, it is made the *unit* of acceleration (thus removing the cumbersome string of decimals which we identified as a problem with the metric system's choice of lengths). The basic unit of acceleration is, therefore, $1 Gf/Tm^2$, which is called the Gee. The length, the Grafut, is a short foot, the zeniGrafut a short inch. The utility of this length is clear. The author, for example, is approximately 6;23 Grafuts tall. This means that he is approximately 62;3 zeniGrafuts tall. It also means that he is about six gravity feet, two gravity inches tall, with $\frac{1}{4}$ of a gravity inch left over (the 0;3 from the zeniGrafut figure). All in all, the Grafut is an intensely useful measure; it is useful scientifically, because it is scientifically derived, yet it is also useful practically and on a daily basis, because it's a convenient length that conforms closely to the varying but approximately similar "foot" units used throughout much of the West for countless generations.

Interestingly enough, TGM also includes a standardized measure of *area*, the Surf, which consists of a single square Grafut. It is worth mentioning that a zenaSurf is approximately equal to one square yard, a measurement of proven utility. Furthermore, a quedraSurf is equal to about half an acre, another very conveniently sized measurement. Further, there is the unit of the Volm, which is the standard unit of volume, which is one cubic Grafut. Three duniVolms (0;03 Volms) is nineteen ounces, just between the American (sixteen ounces) and the British (twenty ounces) pints. This is an extremely convenient measure for everyday use; Pendlebury even suggests coining a new word for it, the "tumblol," to be used for fluids like milk and beer.

Finally, there is the Maz, the unit of mass. Mass is, of course, the amount of matter that an object has; this is not the same as weight, which is the amount that gravity is currently pulling on an object. In TGM, however, whenever one is at earth normal gravity, weight and mass will be the same. So the basic unit of mass, the Maz, is the one that will most often concern us. It is defined as the mass of one Volm of pure air-free water under one standard atmosphere of pressure and at the temperature of maximum density. As a practical matter, this is equal to about 57 (decimal) pounds, or nearly 26 (decimal) kilograms. This makes the zeniMaz about 4;9 pounds.

Further explication of Pendlebury's amazingly precise and well-designed system is outside the scope of this little book. However, the reader is encouraged to read it in its entirety, which is available for free at the Dozenal Society of Great Britain's website.[24]

Clearly, the loss of the metric system is little loss indeed, when such an admirable alternative exists. There is, therefore, nothing in its loss to prevent the adoption of the dozenal system of numeration, which shows itself superior both mathematically and in its popular corresponding system of measurement.

6 Conclusion

It has been a long and fascinating journey. We have determined what number is; what different types of number there are; how different types of number can be written; what qualities we should look for in a base for place notation; what the best practical base for place notation is; and even a new system of mensuration based on the best practical base. It is the author's sincere hope

[24] http://www.dozenalsociety.org.uk/pdfs/TGMbooklet.pdf.

that the reader has enjoyed and profited from this voyage into mathematics as he has.

Mathematical education, particularly in America, has been increasingly neglected in recent years. However, mathematics is still an extremely important part of daily life. One need not be the stereotypical math geek with taped-up eyeglasses and a pocket protector to take an interest in this vitally necessary topic. Indeed, one would have to be extremely derelict *not* to be at least slightly engaged in mathematical issues. It is this principle that made the author interested in mathematics, and it is this principle that once drove, and ought to drive again, mathematical education in our society.

Dozenalism is that principle taken one step further. Namely, that the improvement of a subject of such universal interest is also of universal interest. Dozenalism would be an enormous improvement in our study and use of mathematics as well as in our metrology. Let us, therefore, count in dozens as much as we can. We could make no greater contribution to the mathematical field in our time.

Appendices

Historical Statements on Dozenalism

"Decimal numbering was set up through the invention of man, and a rather poor one at that, not from a necessity of nature as is commonly supposed . . . another system for example, the duodenary, would be very acceptable."[25]

"Decimal arithmetic is a contrivance of man for computing numbers; and not a property of time, space, or matter. Nature has no partialities for the number ten: and the attempt to shackle her freedom with them, will forever prove abortive."[26]

"The duodecimal tables are easy to master, easier than the decimal ones; and in elementary teaching they would be so much more interesting, since young children would find more fascinating things to do with twelve rods or

[25]Blaise Pascal, *De Numeris Multiplicibus*, quoted in Tom Pendlebury, *TGM: A coherent dozenal metrology based on Time Gravity & Mass* (original printed edition, inside cover).

[26]John Quincy Adams, Report of John Quincy Adams in Charles Davies, The Metric System 204 (New York: A. S. Barnes and Company, 10ƐƐ), available at http://books.google.com.

blocks than with ten. Anyone having these tables at command will do these calculations more than one-and-a-half times as fast in the duodecimal scale as in the decimal. This is my experience; I am certain that even more so it would be the experience of others. . . . But the final quantitative advantage, in my own experience, is this: in varied and extensive calculations of an ordinary and not unduly complicated kind, carried out over many years, I come to the conclusion that the efficiency of the decimal system might be rated at about 65 or less, if we assign 100 to the duodecimal."[27]

"[This] plan would teach people to count duodecimally with two new digits . . . and this by itself would recommend it, as duodecimal arithmetic is a coming reform."[28]

"Twelve as a dividend has always been preferred to ten. I can understand the twelfth part of an inch, but not the thousandth part of a metre."[29]

"[Sir Isaac Pitman] sought to make twelve, instead of ten, the basis of computation. He would count and compute by dozens and grosses, instead of by tens and hundreds, and he framed a scheme of nomenclature for weights and measures in accord with the duodecimal unit. The duodecimal scale of reckoning he asserted to be the one that furnished the easiest and most natural system of money, weights and measures. . . . Twelve, he argued, was more completely divisible than ten, in that it can be divided by 2, 3, 4, and 6 without fractional parts. . . . We cannot divide or fold a sheet of printing paper, for a book, in tens, but can readily do so in twelves . . . Isaac Pitman's duodecimal system required two new figures for 10 and 11, and after many experiments he selected ↊ for 10 and ↋ for 11. . . He advocated the adoption of the scheme in the Phonetic Journal, which was paged in accordance with this scheme. He kept his private accounts; and the account of the Phonetic Journal Fund, given in the pages of the Phonetic Journal, were in accord with the new method. He seemed for years almost as hopeful of the adoption of the duodecimal scheme as of the success of the

[27] A. C. Aitken, *The Case Against Decimalization* (Edinburgh / London: Oliver & Boyd, 1176), cited in http://en.wikipedia.org/wiki/Duodecimal, available at http://www.-dozenalsociety.org.uk/pdfs/aitken.pdf. Mr. Aitken was a famous New Zealand mathematician and a so-called "mental calculator" of extraordinary ability.

[28] George Bernard Shaw, Letter to Velizar Godjevatz, printed in F. Emerson Andrews, *My Love Affair with Dozens* in Michigan Quarterly Review XI:2 (1184), available at http://www.dozenal.org/files/E3a My love affair.pdf.

[29] Napoleon Bonaparte, quoted in A. C. Aitken, *The Case Against Decimalization*, *supra* note 27, at 1.

Writing and Spelling reform; and of its ultimate general acceptance and use, he entertained no doubt. The 'three R's, reading, riting, and reckoning,' he urged, would then become so easy and natural that their acquisition would indeed 'come by nature.' . . . [He] never abandoned his conviction that the duodecimal system was the one most worthy of adoption . . . In July, 1896, only a few months before his death, he says in the *Speller*: '. . . reckoning and writing by dozens instead of by tens; then elementary education will become "child's play." My hope for the reckoning reform, counting by dozens instead of tens, has been quickened . . . ' "[2↊]

"Practical reformers . . . have been in general agreement that *twelve* would have been a better base than ten, since it has divisors 2, 3, 4, and 6, a fact that would have made work with fractions easier than it is with base ten (divisors 2 and 5). The learning of only two additional symbols would be worthwhile, compared to the tremendous saving in other arithmetic effort. Charles XII of Sweden was supposed to have been contemplating, at the time of his death, the abolition of the decimal system in all his dominions, in favor of the duodecimal."[2↋]

"Had the number of fingers and toes been different in man, then the prevalent number-systems of the world would have been different also. We are safe in saying that had one more finger sprouted from each human hand, making twelve fingers in all, then the numerical scale adopted by civilized nations would not be the decimal, but the duodecimal. Two more symbols would be necessary to represent 10 and 11, respectively. As far as arithmetic is concerned, it is certainly to be regretted that a sixth finger did not appear. Except for the necessity of using two more signs or numerals and of being obliged to learn the multiplication table as far as 12 x 12, the duodecimal system is decidedly superior to the decimal. The number twelve has for its exact divisors 2, 3, 4, 6, while ten has only 2 and 5. In ordinary business affairs, the fractions $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, are used extensively, and it is very convenient to have a base which is an exact multiple of 2, 3, and 4. Among the most zealous advocates of the duodecimal scale was Charles XII. of Sweden, who, at the time of his death, was contemplating the change for his dominions from the decimal to the duodecimal."[30]

[2↊] Benn Pitman, Sir Isaac Pitman: His Life and Labors 187–90 (C. J. Krehbiel & Co.: Cincinnati, OH, 1126), available at http://books.google.com.

[2↋] Edna Ernestine Kramer, The Nature and Growth of Modern Mathematics 13 (Princeton Paperbacks: Princeton, NJ, 1192).

[30] Florian Cajori, A History of Elementary Mathematics 2–3 (Cosimo, Inc.:

"Of all numbers upon which a system could be based, 12 seems to combine in itself the greatest number of advantages. It is capable of division by 2, 3, 4, and 6, and hence admits of the taking of halves, thirds, quarters, and sixths of itself without the introduction of fractions in the result. From a commercial stand-point this advantage is very great; so great that many have seriously advocated the entire abolition of the decimal scale, and the substitution of the duodecimal in its stead. It is said that Charles XII. of Sweden was actually contemplating such a change in his dominions at the time of his death. In pursuance of this idea, some writers have gone so far as to suggest symbols for 10 and 11, and to recast our entire numeral nomenclature to conform to the duodecimal base. Were such a change made, we should express the first nine numbers as at present, 10 and 11 by new, single symbols, and 12 by 10. From this point the progression would be regular, as in the decimal scale—only the same combination of figures in the different scales would mean very different things. Thus, 17 in the decimal scale would become 15 in the duodecimal; 144 in the decimal would become 100 in the duodecimal; and 1728, the cube of the new base, would of course be represented by the figures 1000. . . . The duodecimal . . . is a system which is called into being long after the complete development of one of the natural systems, solely because of the simple and familiar fractions into which its base is divided. It is the scale of civilization."[31]

"[I]n the duodenary scale, we must have two additional characters for representing 10 and 11, and as these characters may be assumed at pleasure, we shall, in what follows, express 10 by the symbol ϕ, and 11 by π . . . [I]t is evident, as it is indeed from the nature of the subject under investigation, that the greater the radix [base] is, the less will be the number of digits necessary for expressing any given number; but the operations of multiplication, division, &c., will be the more complex; and, therefore, in judging of the advantages and disadvantages of different systems, we ought to keep both these circumstances in view, as also a third, which is the number of prime divisors of the radix; and, on a just estimate of the whole, the radix 12 will be found preferable to any of the other systems. . .

"This leads us to the consideration of the duodenary [dozenal] system of

New York, NY, 11Ɛ3). Original published in 1120. Sadly, later parts of this book refer to the "lower races"; the author certainly does not approve of such nonsense.

[31] Levi Leonard Conant, The Number Concept: Its Origin and Development 131–33 (MacMillan and Co.: New York, NY, 114Ɛ), available from Project Gutenberg at http://www.gutenberg.org/files/16449/16449-h/16449-h.htm.

arithmetic, which, while it possesses all the advantages of the senary [base 6], in point of finite fractions, it is superior even to the decimal system for simplicity of expression; and the only additional burden to the memory is two characters for representing 10 and 11, for the multiplication table in our common arithmetic is generally carried as far as 12 times 12, although its natural limit is only 9 times 9, which is a clear proof that the mind is capable of working with the duodenary system, without any inconvenience or embarrassment; and hence, I think, we may conclude, that the choice of the denary [decimal] arithmetic did not proceed from reflection and deliberation, but was the result of some cause operating unseen and unknown on the inventor of our system."[32]

"It seems clear that the Eldar in Middle-Earth, who have, as Samwise remarked, more time at their disposal, reckoned in long periods, and the Quenya word 'yén', often translated 'year' . . . really means 144 of our years. The Eldar preferred to reckon in sixes and twelves as far as possible. A "day" of the sun they called ré and reckoned from sunset to sunset. The yén contained 52,596 days. For ritual rather than practical purposes the Eldar observed a week or enquië of six days; and the yén contained 8,766 of these *enquier*, reckoned continuously throughout the period."[33]

" 'Odd!' said Graham. 'Gaurdian? Council?' Then turning his back on the new comer, he asked in an undertone, 'Why is this man glaring at me? Is he a mesmerist?'

" 'Mesmerist! He is a capillotomist.'

" 'Capillotomist!'

" 'Yes—one of the chief. His yearly fee is sixdoz lions.'

"It sounded sheer nonsense. Graham snatched at the last phrase with an unsteady mind. 'Sixdoz lions?' he said.

" 'Didn't you have lions? I suppose not. You had the old pounds? They are our monetary units.'

" 'But what was that you said—sixdoz?'

" 'Yes. Six dozen, Sire. Of course, even these little things, have altered. You lived in the days of the decimal system, the Arab system—tens, and little hundreds and thousands. We have eleven numerals now. We have

[32] Peter Barlow, An Elementary Investigation of the Theory of Numbers 222, 226, 243–44 (J. Johnson and Co.: London, 106↋), available at http://books.google.-com.

[33] J. R. R. Tolkien, The Lord of the Rings 1080 (Houghton Mifflin Company: New York, NY, 117↊).

single figures for both ten and eleven, two figures for a dozen, and a dozen dozen makes a gross, a great hundred, you know, a dozen gross a dozand, and a dozand dozand a myriad. Very simple?'

" 'I suppose so,' said Graham."[34]

[34] H. G. Wells, The Sleeper Awakes: A Revised Edition of "When the Sleeper Wakes" (Project Gutenberg: 11Ɛ0), available at http://www.gutenberg.org/-files/12163/12163.txt.